Chapter 4: Matter waves

Matter waves

Previous chapter: We saw ways in which light has a particle-like nature. Now let's see how particles can have a *wave-like* nature.

4.2 Properties of Matter Waves

Waves are described by their **wavelength**, **frequency**, and **amplitude** as a function of position and time.

Wave ***functions*** are solutions to wave ***equations***. They give the amplitude of a wave as a function of position (x) and time (t). For example

- $B(x, t)$ – Magnetic field wave function
- $E(x, t)$ – Electric field wave function

1. Wavelength

The wavelength of a matter wave (AKA the de Broglie wavelength) is

$$\lambda = h/p$$

This relationship indicates that a stationary object ($v = 0$ m/s) would cause the wavelength to approach infinity; just one of the reasons that an object cannot be *truly* stationary.

2. Frequency

The frequency of a matter wave is related to its energy

$$f = E/h$$

It is often more convenient to express frequency and wavelength using angular frequency and wave number, respectively

$$\omega = \frac{2\pi}{T} = 2\pi f \qquad k = 2\pi/\lambda$$

The wave number k is considered a **spatial frequency** since it has units of m^{-1} due to the $1/\lambda$.

Example: what is the spatial frequency of a wave with $\lambda = 1$ m?

Solution: the spatial frequency is $k = 2\pi$ rads/m

Another convenient definition is the **reduced Planck constant**

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}\ \mathrm{J \cdot s}$$

With these definitions, we express the fundamental wave-particle relationships as

$$p = \frac{h}{\lambda} = \hbar k$$

$$E = hf = \hbar\omega$$

3. Velocity

It doesn't make much sense to look at wave velocity for a matter wave. We have a formula describing the *wave* speed, but this is not equal to the *particle* speed

$$v_{wave} = f\lambda = E/p$$

Schrodinger Equation

The wave equation for a matter wave is the Schrodinger equation, which describes the energy and momentum of a particle in time. There are two versions of interest: (1) the **time-dependent** equation and (2) the **time-independent** equation. The following notation is used to distinguish between the two

$$\Psi(x,t) - \text{time-dependent}$$
$$\psi(x) - \text{time-independent}$$

Free-particle Schrödinger equation

A form of the Schrödinger equation that describes a matter wave in the absence of external forces. Note that it contains imaginary number i. This does not mean that the matter wave isn't *real*. Matter waves simply can't be represented by a *single* real function in the same way as an electromagnetic wave. Thus, the complex number representation helps us combine two equations in one.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

Probability Density

The probability of detecting a particle is proportional to the square of the wave's amplitude. Since matter waves are represented by complex functions $\Psi(x,t)$, we find the probability density by multiplying $\Psi(x,t)$ by its complex conjugate $\Psi^*(x,t)$

$$[\mathrm{Re}\ \Psi(x,t)]^2 + [\mathrm{Im}\ \Psi(x,t)]^2 = \Psi^*(x,t)\Psi(x,t) = |\Psi(x,t)|^2$$

$$\text{probability density } = |\Psi(x,t)|^2$$

The Plane Wave
For analysis, we consider the simplest possible solution to the Schrödinger equation. The plane wave solution is the complex exponential

$$\Psi(x,t) = Ae^{i(kx-\omega t)}$$

To verify that this is a solution to the free-particle Schrodinger equation, we must ask

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(Ae^{i(kx-\omega t)}\right) = i\hbar\frac{\partial}{\partial t}\left(Ae^{i(kx-\omega t)}\right)?$$

Taking partial derivatives on both sides, we have

$$\begin{aligned}
-\frac{\hbar^2}{2m}(ik)^2 Ae^{i(kx-\omega t)} &= i\hbar(-i\omega)Ae^{i(kx-\omega t)} \\
-i^2\frac{\hbar^2}{2m}k^2 Ae^{i(kx-\omega t)} &= -i^2\hbar\omega\, Ae^{i(kx-\omega t)} \\
\frac{\hbar^2}{2m}k^2 Ae^{i(kx-\omega t)} &= \hbar\omega\, Ae^{i(kx-\omega t)} \\
\frac{\hbar^2 k^2}{2m} &= \hbar\omega
\end{aligned}$$

Therefore, as long as k and ω have the relation above, the plane wave solution holds for all possible values of x and t. Recognizing that this equation contains $\hbar$, ω, and k, we can insert the fundamental wave-particle relationships

$$\begin{aligned}
\frac{(\hbar k)^2}{2m} &= \hbar\omega \\
\frac{p^2}{2m} &= E
\end{aligned}$$

since $p^2/2m$ is just another way to write kinetic energy, the equation above tells us that the particle's energy is equal (only) to its kinetic energy. Thus, we say that the *Schrödinger equation is related to a classical accounting of energy*.

4.4 Uncertainty Principle

It is impossible to precisely know both the **momentum** and **position** of a particle along the x-axis. The product of the momentum uncertainty and position uncertainty has a strict lower limit:

momentum-position uncertainty

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}$$

4.7 Fourier Transform

The plane wave solution is not a very good model of a real particle. A much better approximation is a **wave pulse** as shown below. We can build such a wave pulse from a sum of pure sine waves.

The waves required to build this wave pulse have wave numbers k, where k is not restricted to integral multiples of some fundamental frequency k_0. In other words, we need to sum over a continuum of wave numbers

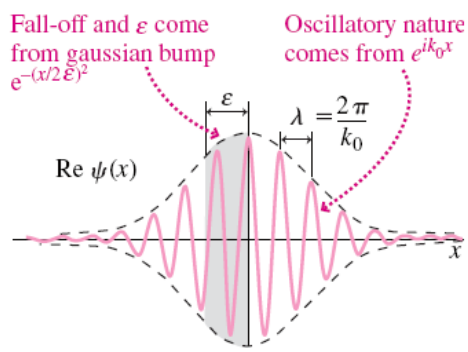
$$\psi(x) = \int_{-\infty}^{+\infty} A(k) e^{ikx} dk$$

But we are interested in $A(k)$ because it tells us how much of each wave number goes into the sum

$$A(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(x) e^{-ikx} dx$$

Gaussian Wave packet

A good approximation for a resonably well-localized particle is the Gaussian wave packet:



$$\psi(x) = Ce^{-(x/2\varepsilon)^2} e^{ik_0x}$$

where C – determines the height
ε – determines the "spread"