Tables of transforms

Jan Molina

Transform	Notation
Laplace	$X(s)$
Fourier Transform (CTFT)	$X(\omega)$ or $X(\Omega)$
Discrete-Time Fourier Transform (DTFT)	$X(e^{j\omega})$
Discrete Fourier Transform (DFT)	X_k
Z-transform	$X(z)$

Definition of Laplace Transform

$$X(s) = \mathcal{L}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Definition of Inverse Laplace Transform

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

Laplace Transform common pairs

Input $x(t)$		Output $X(s)$	ROC
$\delta(t)$	$\leftrightarrow$	1	whole s-plane
$u(t)$	$\leftrightarrow$	$\frac{1}{s}$	$\mathcal{R}e[s] > 0$
$t \cdot u(t)$	$\leftrightarrow$	$\frac{1}{s^2}$	$\mathcal{R}e[s] > 0$
$e^{-at}u(t), a > 0$	$\leftrightarrow$	$\frac{1}{s+a}$	$\mathcal{R}e[s] > -a$
$\cos(\omega t)u(t)$	$\leftrightarrow$	$\frac{s}{s^2+\omega^2}$	$\mathcal{R}e[s] > 0$
$\sin(\omega t)u(t)$	$\leftrightarrow$	$\frac{\omega}{s^2+\omega^2}$	$\mathcal{R}e[s] > 0$
$e^{-at}\cos(\omega t)u(t),\ a > 0$	$\leftrightarrow$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\mathcal{R}e[s] > -a$
$e^{-at}\sin(\omega t)u(t),\ a > 0$	$\leftrightarrow$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\mathcal{R}e[s] > -a$
te^{-at}	$\leftrightarrow$	$\frac{1}{(s+a)^2}$	$\mathcal{R}e[s] > -a$
$1 - e^{-at}$	$\leftrightarrow$	$\frac{a}{s(s+a)}$	$\mathcal{R}e[s] > -a$
t^2	$\leftrightarrow$	$\frac{2}{s^3}$	$\mathcal{R}e[s] > 0$
t^N	$\leftrightarrow$	$\frac{N!}{s^{N+1}}$	$\mathcal{R}e[s] > 0$

Causal vs anti-causal Laplace Transforms

Causal	Anti-causal
$X_c(s) = \mathcal{L}[x_c(t)]$	$X_{ac}(s) = \mathcal{L}[x_{ac}(-t)]\|_{-s}$
Nothing special.	That is, to find an anti-causal Laplace transform of a signal, replace t with $-t$, compute the Laplace transform as normal, then replace every s with $-s$ in the expression.

Laplace Transform properties

Causal functions and constants	$\alpha x(t), \beta y(t)$	$\alpha X(s), \beta Y(s)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(s) + \beta Y(s)$
Right shift in time	$x(t-\alpha)u(t-\alpha)$	$e^{-\alpha s}X(s)$
Frequency shifting	$e^{\alpha t}x(t)$	$X(s-\alpha)$
Convolution in the time-domain	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$
Convolution in the s-domain	$x_1(t)x_2(t)$	$X_1(s) * X_2(s)$
Multiplication by t	$tx(t)$	$-\frac{dX(s)}{ds}$
Multiplication by t^n	$t^n x(t)$	$(-1)^n \frac{d^n X(s)}{ds^n}$
Division by t	$\frac{1}{t}x(t)$	$\int_s^\infty X(u)du$
Multiplication by a Sinusoid	$x(t)\cos(\omega t)$	$\frac{1}{2}[X(s+j\omega) + X(s-j\omega)]$
	$x(t)\sin(\omega t)$	$\frac{j}{2}[X(s+j\omega) - X(s-j\omega)]$
1st Derivative	$\frac{dx(t)}{dt}$	$sX(s) - x(0^-)$
2nd Derivative	$\frac{d^2x(t)}{dt^2}$	$s^2X(s) - sx(0^-) - x'(0)$
Nth Derivative	$\frac{d^N x(t)}{dt^N}$	$s^N X(s) - \sum_{k=0}^{N-1} x^{(k)}(0-)s^{N-1-k}$
Integral of a causal signal	$\int_{0-}^{t} x(\lambda)\, d\lambda$	$\frac{X(s)}{s}$
Expansion/contraction	$x(\alpha t), \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{s}{\alpha}\right)$
Initial value theorem	$x(0) = \lim_{s\to\infty} s \cdot X(s)$	
Final value theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s \cdot X(s)$	

Definition of Fourier Transform

$$X(\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Definition of Inverse Fourier Transform

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Fourier Transform common pairs

Input $x(t)$		Output $X(\omega)$
$\delta(t)$	$\leftrightarrow$	1
$\delta(t-\tau)$	$\leftrightarrow$	$e^{-j\omega\tau}$
$u(t)$	$\leftrightarrow$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$u(-t)$	$\leftrightarrow$	$\frac{-1}{j\omega} + \pi\delta(\omega)$
$\text{sign}\ (t) = 2[u(t) - 0.5]$	$\leftrightarrow$	$\frac{2}{j\omega}$
$A, -\infty < t < \infty$	$\leftrightarrow$	$2\pi A\delta(\omega)$
$Ae^{-at}u(t), a > 0$	$\leftrightarrow$	$\frac{A}{j\omega + a}$
$Ate^{-at}u(t), a > 0$	$\leftrightarrow$	$\frac{A}{(j\omega + a)^2}$
$e^{-a\|t\|}, a > 0$	$\leftrightarrow$	$\frac{2a}{a^2 + \omega^2}$
$\cos(\omega_0 t),\ -\infty < t < \infty$	$\leftrightarrow$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin(\omega_0 t),\ -\infty < t < \infty$	$\leftrightarrow$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$p(t) = A[u(t+\tau) - u(t-\tau)], \tau > 0$	$\leftrightarrow$	$2A\tau\frac{\sin\ (\omega\tau)}{(\omega\tau)}$
$\frac{\sin\ (\omega_0 t)}{\pi t}$	$\leftrightarrow$	$P(\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$
$x(t)\cos\ (\omega_0 t)$	$\leftrightarrow$	$0.5[X(\omega - \omega_0) + X(\omega + \omega_0)]$

Fourier Transform Properties

Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\omega), Y(\omega), Z(\omega)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\omega) + \beta Y(\omega)$
Expansion/contraction in time	$x(\alpha t), \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{\omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\omega)$
Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_X = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{d^n x(t)}{dt^n}, n \geq 1$	$(j\omega)^n X(\omega)$
Frequency differentiation	$-jtx(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
Time shifting	$x(t - \alpha)$	$e^{-j\alpha\omega} X(\omega)$
Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
Modulation	$x(t)\cos(\omega_c t)$	$0.5[X(\omega - \omega_c) + X(\omega + \omega_c)]$
Periodic signals	$x(t) = \sum_k X_k e^{jk\omega_0 t}$	$X(\omega) = \sum_k 2\pi X_k \delta(\omega - k\omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\angle X(\omega) = -\angle X(-\omega)$
Convolution in time	$z(t) = [x^* y](t)$	$Z(\omega) = X(\omega)Y(\omega)$
Windowing/Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\omega)$
Cosine transform	$x(t)$ even	$X(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt$, real
Sine transform	$x(t)$ odd	$X(\omega) = -j\int_{-\infty}^{\infty} x(t)\sin(\omega t)\,dt$, imaginary

Definition of Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$X(\text{z}) = \sum_{n=0}^{N} x[n] \cdot (z^{-1})^{n}$$

(If x[n] has finite support: 0 to N)

Definition of Inverse Z-transform

$$x[n] = x[n] = \frac{1}{j2\pi} \int X(z) \cdot z^{n-1} dz$$

Z-Transform common pairs

Input $x[n]$		Output $X(z)$	ROC
$\delta[n]$	$\leftrightarrow$	1	Whole z-plane
$u[n]$	$\leftrightarrow$	$\frac{1}{1-z^{-1}}$,	$\lvert z\rvert > 1$
$nu[n]$	$\leftrightarrow$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
$n^2u[n]$	$\leftrightarrow$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert > 1$
$\alpha^n u[n], \quad \lvert\alpha\rvert < 1$	$\leftrightarrow$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n u[n], \quad \lvert\alpha\rvert < 1$	$\leftrightarrow$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})}$	$\lvert z\rvert > \lvert\alpha\rvert$
$\cos(\omega_0 n)u[n]$	$\leftrightarrow$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\leftrightarrow$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$\alpha^n\cos(\omega_0 n)u[n], \lvert\alpha\rvert < 1$	$\leftrightarrow$	$\frac{1-\alpha\cos(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2z^{-2}}$	$\lvert z\rvert > 1$
$\alpha^n\sin(\omega_0 n)u[n], \lvert\alpha\rvert < 1$	$\leftrightarrow$	$\frac{\alpha\sin(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2z^{-2}}$	$\lvert z\rvert > \lvert\alpha\rvert$

Z-Transform properties

1. Right shift in time of $x[n] \cdot u[n]$	$x[n] \cdot u[n-q]$	$z^{-q} \cdot X(z)$
2. Right shift of $x[n]$	$x[n-1]$	$z^{-1} \cdot X(z) + x[-1]$
	$x[n-2]$	$z^{-2} \cdot X(z) + x[-2] + z^{-1} \cdot x[-1]$
	$x[n-q]$	$z^{-q} \cdot X(z) + x[-q] + z^{-1} \cdot x[-\mathrm{q}+1] + \cdots + z^{-q+1} \cdot x[-1]$
		$z^{-q} \cdot X(z)$ (if $x[n] = 0$ for negative n)
3. Left shift in time	$x[n+1]$	$z \cdot X(z) - x[0] \cdot z$
	$x[n+2]$	$z^2 \cdot X(z) - x[0] \cdot z^2 - x[1] \cdot z$
4. Multiplication by n	$n \cdot x[n]$	$-z \cdot \frac{dX(z)}{dz}$
	$n^2 \cdot x[n]$	$z \cdot \frac{dX(z)}{dz} + z^2 \frac{d^2X(z)}{dz^2}$
5. Multiplication by a^n	$a^n \cdot x[n]$	$X\left(\frac{z}{a}\right)$
6. Multiplication by $\cos(\omega n)$ and $\sin(\omega n)$	$\cos(\omega n) \cdot x[n]$	$\frac{1}{2}\left[X\left(e^{j\omega} \cdot z\right) + X\left(e^{-j\omega} \cdot z\right)\right]$
	$\sin(\omega n) \cdot x[n]$	$\frac{j}{2}\left[X\left(e^{j\omega} \cdot z\right) - X\left(e^{-j\omega} \cdot z\right)\right]$
7. Summation $x[n] = 0$ for $n = -1, -2, -3, \ldots$ i.e., a causal signal.	$\sum_{i=0}^{n} x[i]$	$\frac{z}{z-1} \cdot X(z)$
8. Convolution	$x[n] * v[n]$	$X(z) \cdot V(z)$

DTFT common pairs

Input $x[n]$		Output $X(e^{j\omega})$ periodic of period 2π	ROC
$\delta[n]$	$\leftrightarrow$	1	$-\pi \leq \omega < \pi$
A	$\leftrightarrow$	$2\pi A\delta(\omega)$	$-\pi \leq \omega < \pi$
$e^{j\omega_0 n}$	$\leftrightarrow$	$2\pi\delta(w - \omega_0)$	$-\pi \leq \omega < \pi$
$\alpha^n u[n], \lvert\alpha\rvert < 1$	$\leftrightarrow$	$\frac{1}{1 - \alpha e^{-j\omega}}$	$-\pi \leq \omega < \pi$
$n\alpha^n u[n], \lvert\alpha\rvert < 1$	$\leftrightarrow$	$\frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$	$-\pi \leq \omega < \pi$
$\cos(\omega_0 n)u[n]$	$\leftrightarrow$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$-\pi \leq \omega < \pi$
$\sin(\omega_0 n)u[n]$	$\leftrightarrow$	$-j\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$-\pi \leq \omega < \pi$
$\alpha^{\lvert n\rvert}, \lvert\alpha\rvert < 1$	$\leftrightarrow$	$\frac{1 - \alpha^2}{1 - 2\alpha\cos(\omega) + \alpha^2}$	$-\pi \leq \omega < \pi$
$p[n] = u[n + N/2] - u[n - N/2]$	$\leftrightarrow$	$\frac{\sin(\omega(N+1)/2)}{\sin(\omega/2)}$	$-\pi \leq \omega < \pi$
$\alpha^n \cos(\omega_0 n)u[n]$	$\leftrightarrow$	$\frac{1 - \alpha\cos(\omega_0)e^{-j\omega}}{1 - 2\alpha\cos(\omega_0)e^{-j\omega} + \alpha^2 e^{-2j\omega}}$	$-\pi \leq \omega < \pi$
$\alpha^n \sin(\omega_0 n)u[n]$	$\leftrightarrow$	$\frac{\alpha\sin(\omega_0)e^{-j\omega}}{1 - 2\alpha\cos(\omega_0)e^{-j\omega} + \alpha^2 e^{-2j\omega}}$	$-\pi \leq \omega < \pi$

DTFT properties

Z-transform	$x[n], X(z), \lvert z \rvert = 1 \in ROC$	$X(e^{j\omega}) = X(z)\rvert_{z=e^{j\omega}}$
Periodicity	$x[n]$	$X(e^{j\omega}) = X\left(e^{j(\omega+2\pi k)}\right)$, k integer
Linearity	$\alpha x[n] + \beta y[n]$	$\alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
Time-shifting	$x[n-N]$	$e^{-j\omega N} X(e^{j\omega})$
Frequency-shift	$x[n]e^{j\omega_0 n}$	$X\left(e^{j(\omega-\omega_0)}\right)$
Convolution	$(x * y)[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n] \cdot y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) d\theta$
Symmetry	$x[n]$ real-valued	$\left\lvert X(e^{j\omega}) \right\rvert$ even function of ω $\angle X(e^{j\omega})$ odd function of ω
Parseval's relation	$\sum_{n=\infty}^{\infty} \lvert X[n] \rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\lvert X(e^{j\omega}) \right\rvert^2 d\omega$	