The Laplace Transform

1 Definition of the Laplace Transform

The Laplace transform is the most general integral transform used in analysis of **continuous-time** signals and systems. Taking the Laplace transform of a signal $x(t)$ means we throw it in the Laplace transform integral where it is multiplied by a complex exponential and then integrate from $-\infty$ to $+\infty$, thus eliminating the time variable t. In other words

$$\mathcal{L}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{1}$$

Since the integration is with respect to t, the resulting expression ends up being in terms of complex number s, which has the definition

$$s = \sigma + j\omega \tag{2}$$

where σ is the **damping factor** and ω is the **radial frequency**. The Fourier transform–a less general version of the Laplace transform–considers only the frequency part ω. Because the Laplace transform involves integrating over an infinite domain, it's essential to determine where and if this integration converges. This delineated area in the complex s-plane, where convergence occurs, is termed the **region of convergence** (ROC). Understanding this region is crucial for properly applying the Laplace transform and interpreting its results.

The Laplace transform of the signal is often denoted using a capitalized version of the original signal. In this case, we say

$$x(t) \longleftrightarrow X(s)$$

$$\boxed{X(s) = \mathcal{L}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt \quad s \in \text{ROC}} \tag{3}$$

The inverse Laplace transform takes us back to the time domain; this time by integrating with respect to complex variable s

$$\boxed{x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \quad \sigma \in \text{ROC}} \tag{4}$$

Some aspects of continuous-time systems can only be examined effectively using the Laplace transform. Stability, transient behavior, and steady-state responses fall into this category. Therefore, delving into Laplace analysis before Fourier analysis is crucial, as Laplace analysis addresses these specific characteristics, while Fourier analysis primarily focuses on frequency properties.

The above definition is for the **two-sided** Laplace transform since the integration of t is from $-\infty$ to $+\infty$, but most of the time we are interested in the **one-sided** Laplace transform when we deal with causal signals and systems. In these cases, the lower integration limit is simply 0.

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt \quad s \in \text{ROC}$$

1.1 Examples

Taking the Laplace transform using its definition (3) is a bit like taking a derivative using the limit definition. That is, it becomes needlessly complicated when doing anything other than the simplest example, and is often not needed. It is instructive, however, to solve at least a few examples using the definition.

Example 1.1.1: Find the Laplace transform of the unit-step signal

$$x(t) = u(t)$$

Solution: Using the definition (3), and ignoring the ROC for now, we know the Laplace transform is

$$X(s) = \mathcal{L}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

First, we replace $x(t)$ with its definition $u(t)$

$$X(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt$$

Then, noting that $u(t)$ is causal signal, defined as 1 from $t = 0$ onward, we replace $u(t)$ with 1 and change the lower integration limit to 0

$$X(s) = \int_{0}^{\infty} 1 \cdot e^{-st}dt$$

Then, we simply evaluate the integral

$$\begin{aligned} X(s) &= \int_0^\infty e^{-st}dt \\ &= -\frac{1}{s} \cdot e^{-st}\Big|_0^\infty \\ &= -\frac{1}{s}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] \\ &= -\frac{1}{s}[0-1] \\ &= \frac{1}{s} \end{aligned}$$

That is,

$$\boxed{X(s) = \mathcal{L}[u(t)] = \frac{1}{s}}$$

Example 1.1.2: Find the Laplace transform of

$$x(t) = e^{-t}u(t)$$

Solution: Using the definition

$$X(s) = \mathcal{L}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Replace $x(t)$ with its definition

$$X(s) = \int_{-\infty}^{\infty} (e^{-t}u(t))e^{-st}dt$$

Again, since $u(t)$ is a causal signal, defined as 1 from $t = 0$ onward, we replace $u(t)$ with 1 and change the lower integration limit to 0

$$X(s) = \int_{0}^{\infty} e^{-t}e^{-st}dt$$

Then, we combine the exponential terms and factor out t to make the integration simpler.

$$X(s) = \int_{0}^{\infty} e^{-(s+1)t}dt$$

Then, we evaluate the integral

$$\begin{aligned} X(s) &= \int_{0}^{\infty} e^{-(s+1)t}dt \\ &= -\frac{1}{s+1} \cdot e^{-(s+1)t}\Big|_{0}^{\infty} \\ &= -\frac{1}{s+1}\left[\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right] \\ &= -\frac{1}{s+1}[0-1] \\ &= \frac{1}{s+1} \end{aligned}$$

That is,

$$\boxed{X(s) = \mathcal{L}[e^{-t}u(t)] = \frac{1}{s+1}}$$

This result is quite useful and will appear often. We can generalize it by replacing the coefficient of 1 in the exponential with with the variable a, giving us the common Laplace transform pair for any value of a

$$\boxed{e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}}$$

2 Poles and Zeros and Region of Convergence (ROC)

What is the region of convergence anyway? To understand the ROC, we must take a closer look at the complex number s and what we call the "s-plane". Recall that s has the definition

$$s = \sigma + j\omega \tag{2}$$

which we can plot on a 2D plane with σ as the horizontal axis and $j\omega$ as the vertical axis. Figure 1 gives the graphical representation of the s-plane. Within this plane, we can create what are known as pole-zero plots of a Laplace transform to show the region of convergence.

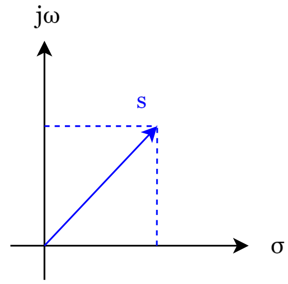

Figure 1: The s-plane

2.1 Poles and Zeros

The Laplace transform always yields some kind of rational function of the form

$$F(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ is the "numerator" polynomial and $D(s)$ is the "denominator" polynomial. To find the poles and zeros, we take each of these functions individually, set them equal to 0, and solve for s. To summarize nicely,

- The **poles** of a function are where $D(s) = 0$ and they are shown as $\times$'s on a pole-zero plot.
- The **zeros** of a function are where $N(s) = 0$ and they are shown as $\circ$'s on a pole-zero plot.

Example 2.1.1: Consider a causal system with Laplace transform $X(s) = \frac{1}{s+1}$. Find the poles.

Solution: Setting the denominator equal to 0

$$\begin{aligned} s + 1 &= 0 \\ s &= -1 \end{aligned}$$

we find that there is only one pole on the real axis at $\sigma = -1$. Since the system is causal, the region of convergence is for all values to the *right* of -1. In other words, $ROC : Re(s) > -1$. The pole-zero plot is given in figure 2. Note that the pole is indicated by a $\times$ symbol on the real axis. The shaded region starting at the pole and continuing to the right is the ROC, which will be discussed in the next section.

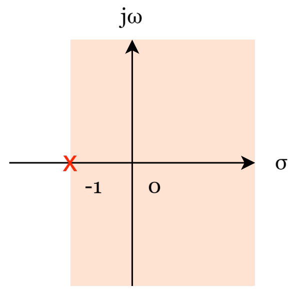

Figure 2: Pole-zero plot for example 2.1.1

Example 2.1.2: An anti-causal signal has Laplace transform $Y(s) = \frac{1}{-s+1}$. Find the poles.

Solution: Setting the denominator equal to 0, we find once again that there is only one pole on the real axis

$$\begin{aligned} -s + 1 &= 0 \\ s &= 1 \end{aligned}$$

This time, because the system is anti-causal, the region of convergence is for all values to the *left* of the pole. In other words, $ROC : Re(s) < 1$. The pole-zero plot is given in figure 3.

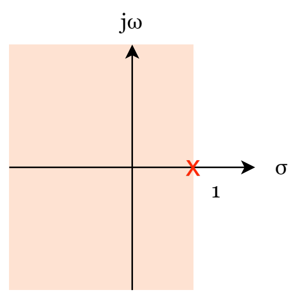

Figure 3: Pole-zero plot for example 2.1.2

The previous examples illustrate what we call “simple” poles, where there is only a real part. With s being a complex number, however, it is possible for the poles to have imaginary parts too; typically complex conjugate pairs.

Example 2.1.3: Find the poles of a causal system with transfer function

$$H(s) = \frac{1}{s^2 + 2s + 2}$$

Solution: Setting the denominator equal to 0, we find

$$\begin{aligned} s^2 + 2s + 2 &= 0 \\ s^2 + 2s + 1 + 1 &= 0 \\ (s+1)^2 + 1 &= 0 \\ (s+1)^2 &= -1 \\ s + 1 &= \pm i \\ s &= -1 \pm i \end{aligned}$$

This time there are two poles located at the same real value of $\sigma = -1$ but with opposite imaginary parts. Much like the first example, the region of convergence covers the whole s-plane for $Re(s) > -1$. In other words, $ROC : Re(s) > -1$. The pole-zero plot is given in figure 4.

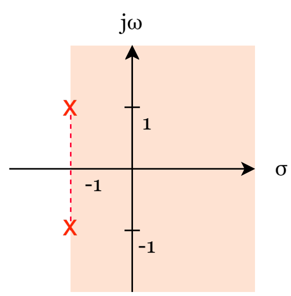

Figure 4: Pole-zero plot for example 2.1.3

Example 2.1.4: Some signals are a combination of a causal and anti-causal part. We refer to these as non-causal signals. Consider the non-causal signal

$$x(t) = e^t u(-t) + e^{-t} u(t)$$

$$\begin{aligned} e^t u(-t) &\implies \text{anti-causal part} \\ e^{-t} u(t) &\implies \text{causal part} \end{aligned}$$

The total Laplace Transform is the sum of the individual Laplace Transforms, i.e.,

$$X(s) = X_{ac}(s) + X_c(s).$$

The causal part has laplace transform

$$X_c(s) = \frac{1}{s+1}$$

with a pole at $\sigma = -1$ and $ROC_1 : Re(s) > -1$. The anti-causal part has laplace transform

$$X_{ac}(s) = \frac{1}{1-s}$$

with a pole at $\sigma = 1$ and $ROC_2 : Re(s) < 1$. Thus, there are two regions of convergence with the overall region of convergence being the intersection $ROC = ROC_1 \cap ROC_2$, therefore

$$ROC : -1 < Re(s) < 1$$

The pole-zero plot is given in figure 5. We note that in this non-causal case, the region of convergence is bounded between the poles.

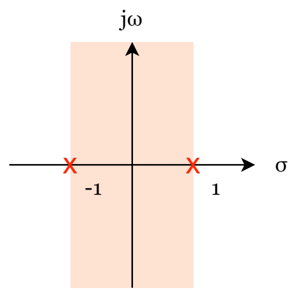

Figure 5: Pole-zero plot for example 2.1.4

What about the zeros? in the previous examples, the numerator was always a constant 1, so there were no zeros to plot. Now, we shall consider a few examples in which where there are zeros (and poles).

Example 2.1.5: Find the poles and zeros of the transfer function

$$H(s) = \frac{3(s+3)}{(s+3)^2 + 64}$$

Solution:

The procedure for finding the poles has not changed. They are

$$s = -3 \pm j8$$

The zeros are found by setting the numerator equal to 0. In this case there is only a single zero

$$s = -3$$

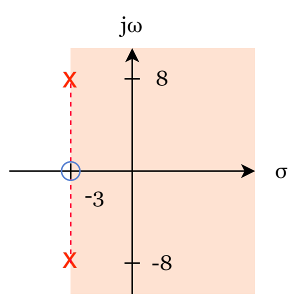

Figure 6: Pole-zero plot for example 2.1.5

Example 2.1.6: Find the poles and zeros of the transfer function

$$H(s) = \frac{s^2 - 2s + 19}{s^3 - 3s^2 + 39s - 37} = \frac{(s-1)^2 + 18}{(s-1)(s^2 - 2s + 37)}$$

Solution:

The zeros are

$$s = 1 \pm j3\sqrt{2}$$

The poles are

$$s = 1 \pm j6$$

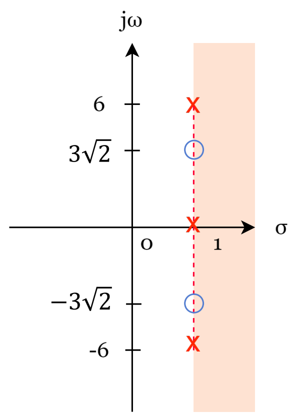

Figure 7: Pole-zero plot for example 2.1.6

2.2 Region of Convergence and Stability

The region of convergence is the area in the s-plane for which the Laplace transform converges. Based on the examples above, it should be clear that the region of convergence is directly related to the location of the poles of a system. In fact, we may say that the region of convergence *begins* at a pole.

As shown in example 2.1.4, if there is more than one region of convergence, then the overall ROC is the intersection of the two: $ROC = ROC_1 \cap ROC_2$.

One of the main purposes of finding the poles is to determine the stability of a system. when we talk about the "stability" of a system, we are talking about the behavior of the system back in the time domain. A system is stable if, under bounded input, its output will converge to some finite value, i.e., the transient terms will eventually vanish. Otherwise, it is unstable.

What do the poles have to do with stability? Put simply, if the poles have a **negative** real part, then the system is stable. If the poles have a **positive** real part, then the system is unstable. The pole-zero plots in figure 8 show the various cases. Plots (a) and (c) correspond to decaying exponentials in the time domain, while plot (b) corresponds to a growing exponential. Plot (d) is a unique case to consider. Although the real part of s is not *positive*, it sits at 0 *and* it has complex conjugate poles, meaning a sinusoidal signal in the time-domain, which does not converge to a finite value so it is *not* stable.

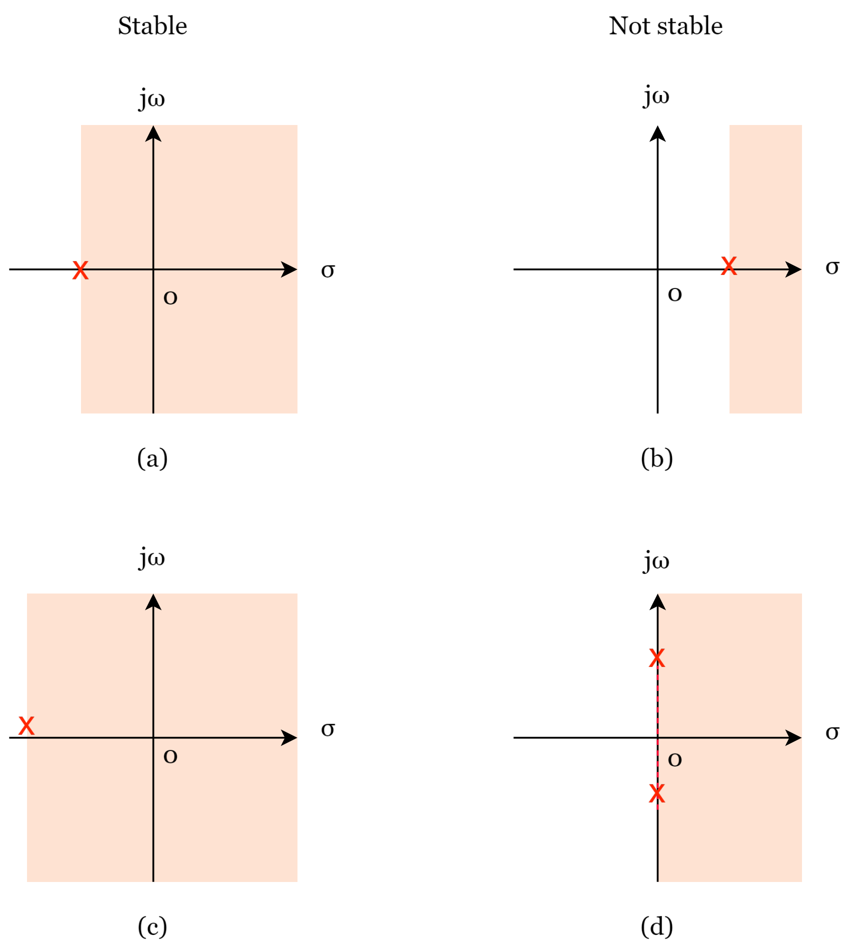

Figure 8: System stability based on location of poles

It is worthwhile to discuss the case of poles at the origin a bit more. When there are poles at the origin, the system may or may not be considered stable depending on the presence of complex conjugate poles. Put simply,

- If $s = 0$, the system is "marginally" stable.
- If $s = 0 \pm$ j(something), the system is unstable.

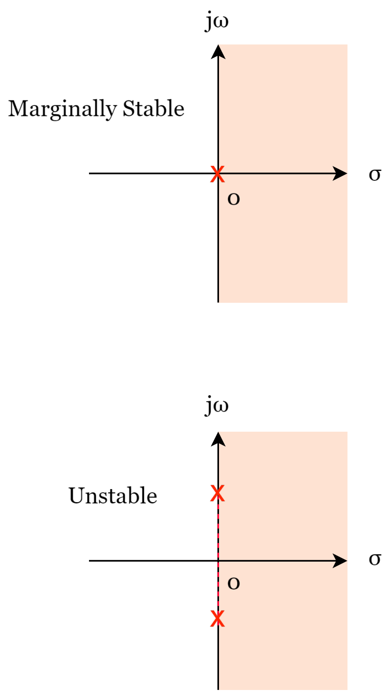

Figure 9: Two cases of poles at the origin