Laplace Transform Examples

1 Basic Laplace transforms

Example 1.1: Find the Laplace transform of the signal

$$x(t) = e^{-t}u(t)$$

Solution: From the table of common Laplace transform pairs, we know that the Laplace transform of a signal of this form is

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}$$

In this case, $a = -1$, therefore

$$\boxed{X(s) = \frac{1}{s+1}}$$

Alternatively, we could think of the $x(t)$ as $u(t)$ with a frequency shift applied due to the multiplication by an exponential term e^{-t}. In that case, you would determine first that the Laplace transform of $u(t)$ is $1/s$, and then apply the frequency shift property (shift amount being $-(-1) = +1$), replacing every s with $s+1$.

Example 1.2: Find the Laplace transform of the signal

$$x(t) = 6e^{-3t}u(t)$$

Solution: From the previous example, we know that the exponential and unit-step part will give us

$$\frac{1}{s+3}$$

and by the linearity of the laplace transform, any constant in front of a signal comes along for the ride. Thus

$$\boxed{X(s) = \frac{6}{s+3}}$$

Example 1.3: Find the Laplace transform of the signal

$$x(t) = u(t) - u(t-1)$$

Solution: As expected, the first term has Laplace transform $1/s$, but the second term is delayed in time so we must apply the time-shifting property. A shift in time corresponds to multiplication by an exponential in the s-domain

$$x(t-a)u(t-a) \longleftrightarrow e^{-as}X(s)$$

In this case, the shift amount a is 1, so

$$\begin{aligned} X(s) &= \mathcal{L}[u(t) - u(t-1)] \\ &= \mathcal{L}[u(t)] - \mathcal{L}[u(t-1)] \\ &= \frac{1}{s} - \frac{1}{s}e^{-(1)s} \\ &= \boxed{\frac{1-e^{-s}}{s}} \end{aligned}$$

Example 1.4: Find the Laplace transform of the signal

$$x(t) = u(t+1) - u(t)$$

Solution: Again, we can apply the time-shifting property, this time with $a = -1$

$$\begin{aligned} X(s) &= \mathcal{L}[u(t+1) - u(t)] \\ &= \mathcal{L}[u(t+1)] - \mathcal{L}[u(t)] \\ &= \frac{1}{s}e^{-(-1)s} - \frac{1}{s} \\ &= \boxed{\frac{e^{s}-1}{s}} \end{aligned}$$

Example 1.5: Find the Laplace transform of the signal

$$x(t) = \cos(\omega t)u(t)$$

Solution: This Laplace transform can be found in most tables already, but we can easily compute it ourselves through a combination of basic properties. First, recall that cosine has the complex definition

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

Letting θ be ωt in the equation above, can rewrite signal $x(t)$ as

$$\begin{aligned} x(t) &= \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)u(t) \\ &= \frac{1}{2}\left[e^{j\omega t}u(t) + e^{-j\omega t}u(t)\right] \end{aligned}$$

Since j and ω are both constants, we can treat them as a single constant $a = j\omega$, leaving us with

$$x(t) = \frac{1}{2}\left[e^{at}u(t) + e^{-at}u(t)\right]$$

This now looks very similar to previous examples. The constant $\frac{1}{2}$ stays out front and we apply the Laplace transform to each term individually

$$\begin{aligned} X(s) &= \frac{1}{2}\mathcal{L}\left[e^{at}u(t) + e^{-at}u(t)\right] \\ &= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \\ &= \frac{1}{2}\left[\frac{s+a}{(s-a)(s+a)} + \frac{(s-a)}{(s+a)(s-a)}\right] \\ &= \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right] \\ &= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] \\ &= \frac{s}{s^2-a^2} \end{aligned}$$

Finally, replacing the a with $j\omega$, we have

$$X(s) = \frac{s}{s^2 - (j\omega)^2}$$

$$\boxed{X(s) = \frac{s}{s^2 + \omega^2}}$$

This is another handy result, and since we did not use any numbers, it is already in its most general form

$$\cos(\omega t)u(t) \longleftrightarrow \frac{s}{s^2+\omega^2}$$

Example 1.6: Find the Laplace transform of the signal

$$x(t) = e^{-10t} \cos(4t) u(t)$$

Solution: First, we note that the negative exponential part is a frequency shift, so we ignore it for now, leaving us with the familar signal

$$\cos(4t)u(t)$$

which has Laplace transform

$$\frac{s}{s^2 + 4^2}$$

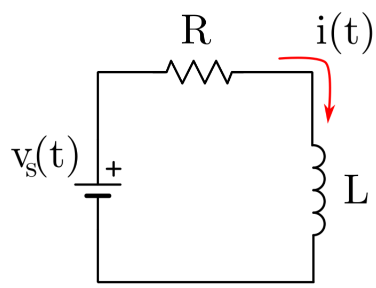
Then, we apply the frequency shift property with shift amount $-(-10) = 10$

$$\boxed{\frac{(s+10)}{(s+10)^2 + 4^2}}$$

2 Simple circuits

The Laplace transform can be used to solve first-order differential equations such as those that arise from the analysis of RC and RL circuits. Finding solutions to differential equations becomes a matter of simple algebraic manipulation in the s-domain and identification of the right inverse Laplace transform to get back to the time-domain.

Example 2.1: Find the impulse response of the RL circuit below. We consider the current $i(t)$ as the output and voltage source $v_s(t)$ as the input of the system in this case. Assume zero initial conditions.



Solution: We want to find the equation for output $i(t)$. Start by applying KVL

$$v_s(t) = Ri(t) + L\frac{di(t)}{dt}$$

Then, since we are looking for the impulse response, we set the input $v_s(t)$ to the unit impulse function $\delta(t)$

$$\delta(t) = Ri(t) + L\frac{di(t)}{dt}$$

Now, we take the Laplace transform of the entire equation and solve for the current

$$\begin{aligned} \mathcal{L}[\delta(t) &= Ri(t) + L\frac{di(t)}{dt}] \\ 1 &= RI(s) + L[sI(s) - i(0)] \end{aligned}$$

Since the initial conditions are 0, the $i(0)$ term goes away

$$\begin{aligned} 1 &= RI(s) + sLI(s) \\ 1 &= I(s)[R + sL] \\ I(s) &= \frac{1}{R + sL} \end{aligned}$$

At this point, we have a familiar-looking result. Dividing the top and bottom by L, we find

$$I(s) = \frac{\frac{1}{L}}{s + \frac{R}{L}}$$

$$I(s) = \frac{1}{L}\frac{1}{s + \frac{R}{L}}$$

$$\boxed{i(t) = \frac{1}{L}e^{-\frac{R}{L}t}u(t)}$$

3 Partial Fraction Expansion

Oftentimes, we must take the inverse Laplace transform of a signal or transfer function that happens to be ratio of polynomials. In such cases, there are no pre-determined results in a table to help us, so we must apply partial fraction expansion to find the solution.

Example 3.1: Find the causal inverse Laplace transform of

$$X(s) = \frac{3s+5}{s^2+3s+2}$$

Solution: Factor the denominator and use partial fraction expansion

$$\begin{aligned} \frac{3s+5}{(s+1)(s+2)} &= \frac{A}{s+1} + \frac{B}{s+2} \\ 3s+5 &= A(s+2) + B(s+1) \\ 3s+5 &= As + 2A + Bs + B \\ 3s+5 &= (A+B)s + (2A+B) \end{aligned}$$

The coefficients of s^n on the lefthand side must be equal to those on the righthand side, that is

$$\begin{aligned} s^2 &: 0 \\ s^1 &: (A+B) = 3 \\ s^0 &: (2A+B) = 5 \end{aligned}$$

Solving for A and B, we find

$$\begin{aligned} A &= 2 \\ B &= 1 \end{aligned}$$

Therefore, the partial fraction expansion becomes

$$X(s) = \frac{2}{s+1} + \frac{1}{s+2}$$

Finally, we take the inverse Laplace transform

$$\boxed{x(t) = \mathcal{L}^{-1}\left[\frac{2}{s+1} + \frac{1}{s+2}\right] = \left[2e^{-t} + e^{-2t}\right]u(t)}$$